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The effect of buoyancy on the boundary layer about a heated horizontal circular cylinder in axial streaming

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The boundary-layer flow over a semi-infinite horizontal circular cylinder heated to a constant temperature and immersed in a uniform axial free stream is discussed in five situations corresponding to successively greater displacements from the leading edge. In the first three cases the drift velocity due to buoyancy is assumed small compared to the axial velocity component. Close to the leading edge of the cylinder the techniques of Seban & Bond are extended to include the drift velocity; far from the leading edge the asymptotic series methods of Stewartson, of Glauert & Lighthill, and of Eshghy & Hornbeck are employed to obtain a solution for the drift velocity. In the intermediate zone where the series solutions do not apply the appropriate partial differential equations are solved numerically. Still further downstream than the region where the 'asymptotic' solutions hold it is assumed that the boundary-layer flow is primarily convective and that the boundary layer is thin compared with the radius of the cylinder. A series solution is obtained which is valid near the lowest generator of the cylinder. Numerical methods are used to advance this solution upwards around the cylinder by solving the full boundary-layer equations step-by-step.

1. Introduction

We consider the velocity and temperature distributions in the boundary layer about a semi-infinite horizontal circular cylinder which is heated to a constant temperature T_w above the ambient temperature T_∞ and is immersed in a uniform axial free stream U_∞ of infinite extent from the boundary-layer point of view. Throughout the discussion it is assumed that $(X/a)(\mathscr{G}/\mathscr{R}^2) \ll 1$ so that the drift (azimuthal) velocity component is small compared with the axial velocity component (here X = distance from the leading edge; a = the radius of the cylinder; $\mathscr{G}$ = the Grashof number $= \beta g(T_w - T_\infty)\, a^3/\nu^2$; $\mathscr{R}$ = the Reynolds number $= aU_\infty/\nu$). Near the leading edge $X/a \ll \mathscr{R}$, corresponding to a boundary layer thin compared with a, and there is no restriction on the size of the Grashof number. Here the well-known series solution of Seban & Bond is extended to include the drift velocity. Further from the leading edge, where the boundary layer is neither thick nor thin compared with the cylinder radius a, the appropriate partial differential equations are solved numerically. Still further downstream the problem is treated subject to $(X/a)(\mathscr{G}/\mathscr{R}^2) \ll 1 \ll (X/a)(1/\mathscr{R})$, which guarantees that

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the boundary layer is thick compared to a and that the drift velocity remains small compared to the axial velocity component. The solution is accomplished by extending the work of Stewartson (1955), of Glauert & Lighthill (1955), and of Eshghy & Hornbeck (1967) to satisfy the boundary conditions on the stream function and the temperature through exponentially small terms in order to obtain a solution for the drift velocity which satisfies the boundary conditions asymptotically. Still further from the leading edge the buoyant fluid will rise about the cylinder and prevent the boundary layer from growing any thicker. Conditions will become independent of X as buoyancy becomes dominant. If $\mathscr{G} \gg 1$ there will be a free-convection boundary layer with a superposed axial flow; this situation is discussed in §§ 6–8.

In each of the three regions considered in §§ 2–5 the three-dimensional problem reduces to a two-dimensional problem because of the assumption that the drift velocity is small compared with the axial velocity component. As a result, only the drift velocity depends upon the azimuthal displacement (measured around the cylinder from the lowest generator), and this variable is easily removed from the equations. Near the leading edge and far downstream the resulting two-dimensional problem is amenable to series solution, the appropriate changes of variable being dictated by the thinness of the boundary layer in the first case and by the thickness of the layer in the latter. In between, where the thickness of the layer is of the order of the radius of the cylinder, the partial differential equations of the two-dimensional problem are solved by an economical numerical method described by Terrill (1960).

In the second half of the paper (§§ 6–8) we discuss the flow as buoyancy becomes dominant. We assume that this is the approach to a steady-state régime in which convection by the drift velocity dominates over convection by the axial velocity: this essentially free-convective behaviour occurs so far from the leading edge of the cylinder that it is in fact independent of distance from the leading edge. Thus the equations of motion reduce to two-dimensional equations.

In addition to a large Reynolds number we assume a large Grashof number, which ensures the boundary layer will be thin compared with the radius of the cylinder. The thinness of the boundary layer suggests a transformation which leads to a series solution valid near the lowest generator. The series furnishes the initial profiles for a step-by-step numerical technique which uses the full boundary-layer equations to obtain profiles at stations successively further from the lowest generator.

2. Equations of motion when axial flow dominant

The assumption that $U^2/\alpha^2 \ll (T_w - T_\infty)/T_\infty \ll 1$ ($\alpha =$ the velocity of sound) permits us to ignore the temperature variation of the thermometric conductivity κ and of the kinematic viscosity ν and to treat the fluid as incompressible, so that changes in density are significant only in producing buoyancy forces.

With these assumptions the boundary-layer equations are

$$RU_X + (RV)_R + RW_\theta = 0, \tag{1}$$

$$UU_X + VU_R + WU_\theta/R = \nu(U_{RR} + U_R/R), \tag{2}$$

$$UT_X + VT_R + WT_\theta/R = \kappa(T_{RR} + T_R/R), \tag{3}$$

$$UW_X + (V/R)(WR)_R + WW_\theta/R = \nu(W_{RR} + W_R/R + (W_{\theta\theta} - W)/R^2) + \beta g(T - T_\infty)\sin\theta. \tag{4}$$

Here R is the radial displacement measured outwards from the axis of the cylinder; the azimuthal angle θ is measured in the clockwise sense with $\theta = 0$ at the lowest generator. U, V, and W are the velocity components in the X, R, and θ directions, respectively, and T is the temperature of the fluid. Subscripts represent partial derivatives.

The choice $W(X, R, \theta) = U_\infty(4X/a)(\mathscr{G}/\mathscr{R}^2)\, w(X, R)\sin\theta$, according to which W grows linearly with X but stays small compared with U because of the assumption $(X/a)(\mathscr{G}/\mathscr{R}^2) \ll 1$, enables us to ignore terms $(W/R)\,\partial/\partial\theta$ and to replace (1)–(4) by

$$RU_X + (RV)_R = 0, \tag{1'}$$

$$UU_X + VU_R = \nu(U_{RR} + U_R/R), \tag{2'}$$

$$UT_X + VT_R = \kappa(T_{RR} + T_R/R), \tag{3'}$$

$$U(w_X + w/X) + (V/R)(wR)_R = \nu(w_{RR} + w_R/R - 2w/R^2) + \beta g(T - T_\infty)/X, \tag{4'}$$

where a factor $X \sin\theta$ has been removed from $W(X, R, \theta)$.

We consider (1′)–(4′) subject to the boundary conditions $U = V = w = 0$, $T = T_w$ at $R = a =$ the radius of the cylinder, $U = U_\infty$, $V = w = 0$, $T = T_\infty$ as $R \to \infty$ or when $X = 0$.

3. The solution near the leading edge

The boundary layer near the leading edge, thin compared with the radius of the cylinder, is formed primarily by viscous retardation of the mainstream.

The analysis for the U and V velocity components and for the temperature has already been carried out by Seban & Bond (1951). With the introduction of similarity variables $x = (4/a)(\nu X/U_\infty)^{\frac{1}{2}}$, $y = (U_\infty/\nu X)^{\frac{1}{2}}(R^2 - a^2)/(4a)$, a stream function $S = a(\nu X U_\infty)^{\frac{1}{2}} f(x, y)$, and a dimensionless temperature $t = (T - T_\infty)/(T_w - T_\infty)$, we satisfy (1′) by defining $RU = S_R$ and $RV = -S_X$, and (2′)–(4′) become

$$[(1 + xy)f'']' + ff'' + x(f_x f'' - f'f_x') = 0, \tag{5}$$

$$(1/\sigma)[(1 + xy)t']' + ft' + x(f_x t' - f't_x) = 0, \tag{6}$$

$$(1 + xy)^2 w'' + w'(1 + xy)(x + xf_x + f) - \tfrac{1}{2}xw(x - xf_x - f + yf') - 2w(1 + xy)f' - x(1 + xy)f'w + t(1 + xy) = 0, \tag{7}$$

where σ is the Prandtl number ν/κ. Here and from now on a prime represents the partial derivative with respect to y in partial differential equations and the ordinary derivative with respect to y in ordinary differential equations.

In the new notation the boundary conditions are $f = f' = f_x = w = 0$, $t = 1$ at $y = 0$, $f' = 2$, $t = w = 0$ as $y \to \infty$.

Equation (6) differs from the corresponding equation of Seban & Bond in that a viscous dissipation term which those authors include is neglected here.

The series expansions

$$f(x,y) = f_0(y) + xf_1(y) + x^2 f_2(y) + \ldots, \quad t(x,y) = t_0(y) + xt_1(y) + x^2 t_2(y) + \ldots,$$
$$w(x,y) = w_0(y) + xw_1(y) + x^2 w_2(y) + \ldots$$

inserted into (5)–(7) yield the equations

$$w_0'' + w_0' f_0 - 2w_0 f_0' + t_0 = 0,$$
$$w_1'' + w_1' f_0 - 3w_1 f_0' + 2yw_0'' + w_0'(1 + 2f_1 + yf_0) + w_0(\tfrac{1}{2}f_0 - \tfrac{5}{2}yf_0' - 2f_1') + t_1 + yt_0 = 0,$$
$$w_2'' + w_2' f_0 - 4w_2 f_0' + w_1'(1 + 2f_1 - yf_0) + w_1(\tfrac{1}{2}f_0 + \tfrac{5}{2}yf_0' - 3f_1') + w_0'(-2y + 3f_2 - 2yf_1 + y^2 f_0) + w_0(-\tfrac{1}{2} + \tfrac{3}{2}f_1' - yf_0 + f_1 - y^2 f_0' - 2f_2') + t_2 - yt_1 + y^2 t_0 = 0,$$

with boundary conditions $w_0 = w_1 = w_2 = 0$ at $y = 0$ and as $y \to \infty$, which we consider in addition to Seban & Bond's analogously obtained ordinary differential equations for f_n', t_n (g_n in their notation).

	$\sigma = \frac{1}{2}$	$\sigma = 2$
$f_0(0)$	0	0
$f_0'(0)$	0	0
$f_0''(0)$	1·32823	1·32823
$t_0(0)$	1	1
$t_0'(0)$	−0·51859	−0·84462
$w_0(0)$	0	0
$w_0'(0)$	0·48332	0·39032
$f_1(0)$	0	0
$f_1'(0)$	0	0
$f_1''(0)$	0·69432	0·69432
$t_1(0)$	0	0
$t_1'(0)$	−0·31909	−0·38319
$w_1(0)$	0	0
$w_1'(0)$	−0·05471	−0·05156
$f_2(0)$	0	0
$f_2'(0)$	0	0
$f_2''(0)$	−0·16414	−0·16414
$t_2(0)$	0	0
$t_2'(0)$	0·07608	0·09308
$w_2(0)$	0	0
$w_2'(0)$	−0·24632	−0·13691

TABLE 1. Initial values for the series of § 3

These ordinary differential equations were solved numerically on Manchester University's Atlas computer and were used to provide a starting profile at $x = 0{\cdot}01$ for the step-by-step numerical solution of (5)–(7), which is discussed in §4. Table 1 gives the initial conditions for $\sigma = \frac{1}{2}$ and 2.

4. Step-by-step continuation of the solution near the leading edge

The series solutions presented in § 3 become awkward when they must be extended further and further to yield the desired accuracy. Instead of continuing the series we use them as they stand to provide initial profiles for a step-by-step numerical solution of (5), (6), and (7) whose accuracy is limited only by the time required to make the necessary calculations on a computer.

The numerical methods used have two distinctive features: equation (5) is treated as a second-order equation by solving for f' and substituting an 'integral' of f' for f itself; the profile $f'(x+dx, y)$ is actually obtained by solving for the quantity $f'(x+dx, y)+f'(x, y)$. The starting approximation is $2f'(x, y)$; once the

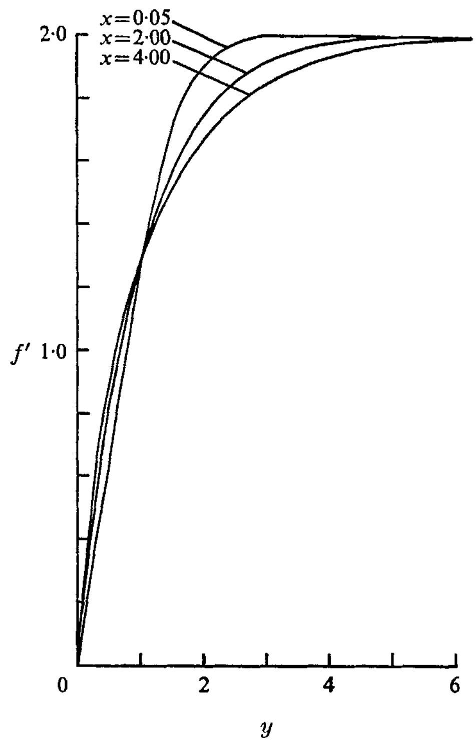

FIGURE 1. Axial velocity profiles.

iterative approximation to $f'(x+dx, y)+f'(x, y)$ is acceptable the 'old' profile $f'(x, y)$ is subtracted, leaving the 'new' profile. Since Terrill's (1960) method requires only minor adaptations for the present problem, it is not necessary to describe it more fully here.

First and second partial derivatives with respect to y were replaced by the standard three-point central-difference approximations; the derivative with respect to x was handled in the same way but cast into such a form that the quantity $1+2x/dx$ became a parameter of the problem, allowing the finite-difference equations to be solved like ordinary finite-difference equations at any given value of x. At any particular point x_k the mth iteratively obtained approximate profile $f'^m(x_k, y)$ was judged to be acceptable if

$$|f'^m(x_k, y_j)-f'^{m-1}(x_k, y_j)| < 10^{-5}, \quad j = 1, 2, \ldots, 140.$$

The mesh-lengths used were $dy = 0{\cdot}05$, $dx =$ variable. The starting point was $x = 0{\cdot}01$ with $dx = 0{\cdot}002$. As the integration proceeded away from the leading edge it was possible to increase dx without slowing down the rate of convergence of the iterations. The integrations were performed twice, in one case with a mesh-length dx and in the other with $2dx$, and the size of the step was kept small enough to guarantee agreement to at least the four decimal places desired.

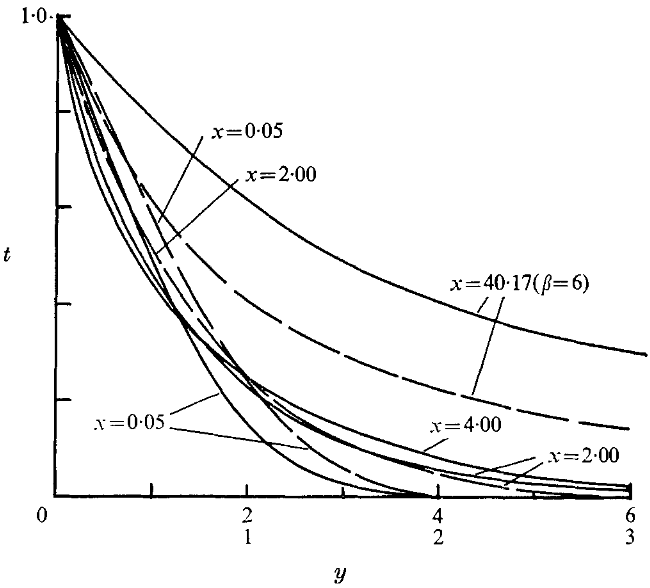

FIGURE 2. Temperature profiles. ——, $\sigma = \frac{1}{2}$; — —, $\sigma = 2$.

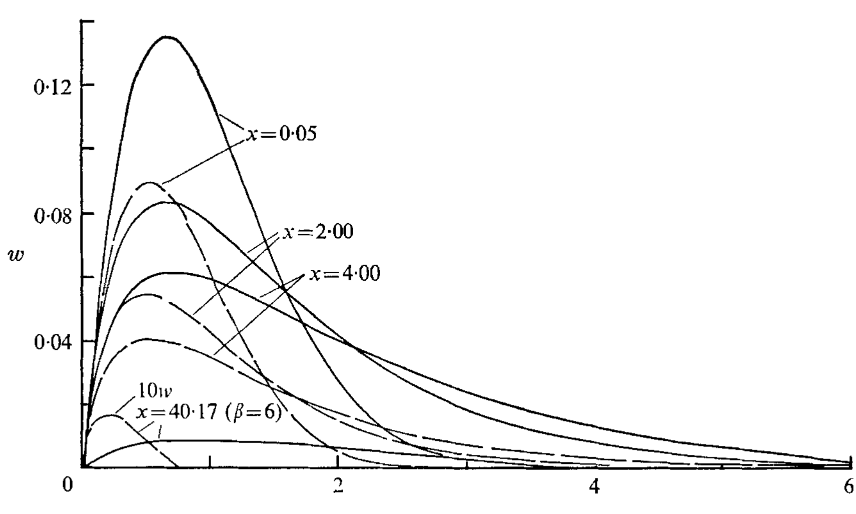

FIGURE 3. Drift velocity profiles. ——, $\sigma = \frac{1}{2}$; — —, $\sigma = 2$.

Figures 1–3 show profiles for f', t, and w for a few values of x ranging from $x = 0{\cdot}05$ to $x = 4$. Figures 2 and 3 also include approximate temperature and drift-velocity profiles for $x = 40{\cdot}17$. These were obtained from the leading terms of the series solutions developed in § 5. Table 2, which is available from the editor, presents the dimensionless flow parameters

$$d1 = \lim_{y\to\infty}(2y-f), \quad d2 = \int_0^\infty f'(1-\tfrac{1}{2}f')\,dy, \quad dt = \int_0^\infty f't\,dy, \quad dw = \int_0^\infty f'w\,dy,$$

and also $f''(x,0)$, $t'(x,0)$ and $w'(x,0)$. The integrals were approximated by the Euler–Maclaurin formula, following Terrill, and the derivatives were obtained by means of a Maclaurin expansion about $y = 0$. The results are given for $\sigma = \frac{1}{2}, 2$.

5. The solution in the Glauert–Lighthill–Stewartson region

Still further from the leading edge we consider the problem subject to

$$(X/a)(\mathscr{G}/\mathscr{R}^2) \ll 1 \ll (X/a)(1/\mathscr{R}),$$

so that the boundary-layer thicknesses are now many times the radius of the cylinder, but we still have $W \ll U, V$.

The analysis for this region parallels Stewartson's analysis for the U and V velocity components in terms of a stream function. Eshghy & Hornbeck's solution for the temperature is extended so that, as in Stewartson's work, the boundary condition on the cylinder is satisfied through exponentially small terms. These refinements permit a solution for the W velocity component which satisfies the boundary condition on the cylinder asymptotically.

The equations in physical variables and the boundary conditions are those given in § 2 (equations (1′)–(4′)). We let

$$\beta = \log(4\nu X/U_\infty a^2), \quad y = U_\infty R^2/4\nu X, \quad S = \nu X f(y,\beta), \quad U = \tfrac{1}{2}U_\infty f',$$
$$V = (\nu/R)(yf'-f-f_\beta), \quad t = 2(T-T_w)/(T_\infty - T_w),$$
$$W = 2U_\infty(\mathscr{G}/\mathscr{R}^2)(X/a)\,yw(y,\beta)\sin\theta.$$

In terms of the new variables the equations of motion are

$$yf''' + [1+\tfrac{1}{2}(f+f_\beta)]f'' - \tfrac{1}{2}f'f'_\beta = 0, \tag{8}$$

$$(1/\sigma)\,yt'' + [1/\sigma + \tfrac{1}{2}(f+f_\beta)]\,t' - \tfrac{1}{2}f't_\beta = 0, \tag{9}$$

$$4y^2w'' + 2yw'(5+f+f_\beta) + 3w(f_\beta+f-yf') = 2yf'w_\beta + t - 2, \tag{10}$$

subject to $f' = f+f_\beta = t = w = 0$ on $y = e^{-\beta}$ and $f' = t = 2$, $w = 0$, as $y \to \infty$.

Since the parameter β is large, we expand $f(y,\beta)$ as follows:

$$f(y,\beta) = 2y + [f_1(y) + (G_1(y)\log\beta + H_1(y))\,e^{-\beta}]/\beta$$
$$+ [f_2(y) + (G_2(y)\log\beta + H_2(y))\,e^{-\beta}]/\beta^2 + \dots.$$

Putting this into (8) and equating coefficients of powers of β yields

$$\left.\begin{aligned}
yf_1''' + f_1''(1+y) &= 0,\\
yf_2''' + f_2''(1+y) + \tfrac{1}{2}f_1''f_1 + f_1' &= 0,\\
yG_1''' + G_1''(1+y) + G_1' &= 0,\\
yG_2''' + G_2''(1+y) + G_2' + \tfrac{1}{2}G_1''f_1 + \tfrac{1}{2}G_1'f_1' + G_1' &= 0,\\
yH_1''' + H_1''(1+y) + H_1' &= 0,\\
yH_2''' + H_2''(1+y) + H_2' + \tfrac{1}{2}H_1''f_1 + \tfrac{1}{2}H_1'f_1' + H_1' - G_1' &= 0.
\end{aligned}\right\} \tag{11}$$

The first two of these equations and their solutions were obtained by Glauert & Lighthill (1955); the rest were obtained by Stewartson (1957).

The solutions of some of (11) are

$$f_1 = 2(yEi(-y)+e^{-y}-1), \quad G_1' = g\,e^{-y}, \quad H_1' = h\,e^{-y},$$

where g and h are arbitrary constants, and the $Ei(-z)$, $EI(-z)$ which appear in the solutions for f_1, f_2, etc., are defined to be

$$Ei(-z) = \int_\infty^z e^{-t}\,dt/t, \quad EI(-z) = \int_\infty^z Ei(-t)\,dt/t.$$

Solving the equations for f_2, H_2 and so on rapidly becomes very tedious, but it is possible to satisfy the boundary conditions without integrating all of these equations explicitly. What concerns us is their behaviour when $y \ll 1$ (i.e. as $y \to e^{-\beta}$), since it is possible to satisfy the conditions as $y \to \infty$ but not those on $y = e^{-\beta}$ by considering the f_n one at a time.

For $0 \leqslant y \ll 1$ we use the approximations

$$Ei(-y) = \gamma + \log y - y, \quad EI(-y) = \pi^2/12 + \tfrac{1}{2}Ei^2(-y)$$

through terms in y; γ = Euler's constant = 0·5772157.... Then we have approximately

$$f_1' = 2(\gamma + \log y - y),$$
$$f_2' = C_2 + 2\gamma \log y + 4y + 2y \log y,$$
$$G_2' = G_1'(\log y - \gamma),$$
$$H_2' = H_1'(\log y - \gamma) - G_1' \log y,$$

where the constants $B_n, C_n, n = 2, 3, 4, \ldots$, defined in Eshghy & Hornbeck, arise in the approximation $f_n' = B_n + C_n \log y$ for $y \ll 1$ and satisfy the relation $C_n = B_{n-1}$.

When we insert these approximations into the series expansion for f' and require $f'(e^{-\beta}, \beta) = 0$, we obtain

$$f' = (G_1' - 4)\,e^{-\beta}/\beta + O(e^{-\beta}/\beta^2).$$

Hence by choosing $G_1' = 4$ on $y = e^{-\beta}$ we can satisfy the boundary condition to $O(e^{-\beta}/\beta)$. Similarly the requirement that $f + f_\beta = 0$ on $y = e^{-\beta}$ implies

$$O = f + f_\beta = e^{-\beta}[C_1 - C_2 + G_1(1 - \log \beta) - H_1]/\beta^2 + \ldots.$$

By taking $H_1 = C_1 - C_2 (= 2(1-\gamma))$ and $G_1 = 0$ on $y = e^{-\beta}$ we can satisfy the condition $f + f_\beta = 0$ on $y = e^{-\beta}$ to $O(e^{-\beta}/\beta^2)$.

In the course of satisfying the boundary conditions we have determined the values of G_1, G_1', and H_1 on $y = e^{-\beta}$, but the corresponding value of H_1' remains undetermined, presumably because this analysis is ambiguous, as Stewartson (1957) noted, since the origin of X displacement is unspecified.

In order to satisfy the boundary conditions on the cylinder more accurately it would be necessary to include higher powers of $\log \beta$ in the series expansion.

The solution of (9) for the dimensionless temperature t follows by the same techniques. Eshghy & Hornbeck's definition, which we use here, is very convenient since when $\sigma = 1$ (9) for t is identical with (8) regarded as an equation for f'. Neglecting the boundary condition on $f+f_\beta$, we see that the conditions on f' itself are identical with those on t. Thus when $\sigma = 1$ we can make a few modifications in f' to obtain the solution for the temperature. Since one boundary condition has been eliminated the complementary functions H_1, H_2 are no longer needed. We make the choice $G_1' = 4$ as before, but G remains unspecified because the analysis does not fix the behaviour of the temperature at the leading edge of the cylinder.

When $\sigma \neq 1$ we let

$$t(y,\beta) = 2+[t_1(y)+T_1(y)\,e^{-\beta}/\beta^k]/\beta+[t_2(y)+T_2(y)\,e^{-\beta}/\beta^k]/\beta^2+\dots,$$

where t_1 and t_2 were found by Eshghy & Hornbeck and we extend the series by including the terms involving T_1 and T_2. We shall choose $k = 1$ when $1/\sigma$ is non-integral and $k = 0$ when $1/\sigma = 2, 3, 4, \dots$.

We want to satisfy (9) and the corresponding boundary conditions. By inserting the expansion for $t(y,\beta)$ into (9) we obtain

$$(1/\sigma)\,yt_1''+(1/\sigma+y)\,t_1' = 0,$$
$$(1/\sigma)\,yt_2''+(1/\sigma+y)\,t_2'+\tfrac{1}{2}f_1t_1'+t_1 = 0,$$
$$yT_1''+(1+y\sigma)\,T_1'+T_1 = 0.$$

The solution for $t_1(y)$ appropriate to $t \to 0$ as $y \to \infty$ is $t_1 = 2Ei(-y\sigma)$.

We can satisfy the boundary conditions term-by-term as $y \to \infty$ but not when $y = e^{-\beta}$, so again we consider the approximate forms of the solutions for $y \ll 1$.

Eshghy & Hornbeck's result $t_n = b_n+c_n\log y, n = 1,2,3,\dots$ for $y \ll 1$ ($c_n = b_{n-1}$, and the constants are given in their article) becomes

$$t_1 = b_1+c_1\log y-2y\sigma,$$
$$t_2 = b_2+c_2\log y+y[4(\gamma+(1-\gamma)\,\sigma)+2(2-\sigma)\,(\log y\sigma)],$$

when terms in y are included. The solution for T_1 which vanishes at infinity is $T_1 = M\,e^{-y\sigma}\mathscr{F}[1-1/\sigma, 1; y\sigma]$; $\mathscr{F}$ is a confluent hypergeometric function and M is an arbitrary constant.

When $1/\sigma$ is not an integer $\mathscr{F}$ is the confluent hypergeometric function U as defined in Slater (1960). $U[a,b;x]$ can be approximated by

$$U[a,b;x] = -(\log x+d(\log\Gamma(a))/da+2\gamma)/\Gamma(a)$$

when $x \ll 1$.

Inserting these approximations for U and for t_1 and t_2 into the series for $t(y,\beta)$ and requiring $t = 0$ on $y = e^{-\beta}$ shows that the choice $M = 4\Gamma(1-1/\sigma)$ for the arbitrary constant satisfies the boundary condition to $O(e^{-\beta}/\beta)$.

When $1/\sigma = 2,3,4,\dots$ we take $\mathscr{F} = {}_1F_1$; this is the 'usual' confluent hypergeometric function and is also defined in Slater. Because $1-1/\sigma$ is now a negative integer, ${}_1F_1$ is a finite polynomial.

When $y \ll 1$ and $1/\sigma =$ an integer N,

$$T_1 = M(1-y+(N+1)\,y^2/(4N)+\dots)$$

approximately. In this case we take $M = 4$ and satisfy the boundary condition to the same order as before.

To improve the accuracy of the solution it would be necessary to introduce into the series additional terms multiplied by $e^{-\beta}\log\beta/\beta^3$ and so on.

To solve (10) for the drift velocity we employ a series

$$w(y,\beta) = [w_1(y)+e^{-\beta}Q_1(y)/\beta^k+\beta^2 e^{-\frac{3}{2}\beta}S_1(y)+\ldots]/\beta$$
$$+[w_2(y)+e^{-\beta}Q_2(y)/\beta^k+\beta^2 e^{-\frac{3}{2}\beta}S_2(y)+\ldots]/\beta^2,$$

where $k = 1$ when $1/\sigma$ is non-integral and $k = 0$ when $1/\sigma = 2, 3, 4, \ldots$.

By inserting this expression into (10) we obtain

$$4y^2w_1''+2yw_1'(5+2y) = t_1,$$
$$4y^2S_1''+2yS_1'(5+2y)+6yS_1 = 0,$$
$$4y^2Q_1''+2yQ_1'(5+2y)+4yQ_1 = T_1,$$
$$4y^2S_2''+2yS_2'(5+2y)+2yS_1'f_1+3S_1f_1+6yS_2-4yS_1 = 0.$$

The solution for S_1 which vanishes at infinity is

$$S = Ky^{-\frac{3}{2}}\int_\infty^y t^{\frac{1}{2}}e^{-t}\,dt, \quad K \text{ arbitrary.}$$

When $y \ll 1$, the solutions of the equations for w_1, w_2, etc. can all be approximated by

$$w_n = A_n\log^2 y+B_n\log y+C_n+y(D_n\log^2 y+E_n\log y+F_n)+\ldots,$$

where, in terms of Eshghy & Hornbeck's b_n, c_n, $A_n = \frac{1}{12}c_n$, and $B_n = \frac{1}{6}b_n-\frac{1}{9}c_n$. The C_n must be determined from the boundary conditions on the cylinder, and $D_1 = 0$.

When $y \ll 1$, the solutions for S_1 and S_2 are approximately

$$S_1 = K(-\tfrac{1}{2}\pi^{\frac{1}{2}}y^{-\frac{3}{2}}+\tfrac{2}{3}+\ldots),$$
$$S_2 = y^{-\frac{3}{2}}\{A'+K\pi^{\frac{1}{2}}(y-y^2+\ldots)\}, \quad A' \text{ arbitrary.}$$

In order to satisfy the condition $w(e^{-\beta},\beta) = 0$ we must choose $K = 2A_1\pi^{-\frac{1}{2}}$, $A' = B_1-A_2$, and $C_{n-1}-B_n+A_{n+1} = 0$, $n \geqslant 2$.

The terms $Q_1e^{-\beta}/\beta^{1+k}$, etc., required in the series for w (since the temperature series includes a term $T_1e^{-\beta}/\beta^{1+k}$) do not provide any arbitrary constants which can be used to satisfy the boundary condition through exponentially small terms. To do this it would be necessary to include terms

$$e^{-\frac{5}{4}\beta}(P_1(y)+P_2(y)/\beta+\ldots)$$

in the series $\left(\text{with } 4y^2P_1''+2yP_1'(5+2y)+10yP_1 = 0 \text{ and } P_1 = \text{const.}\, e^{-y}\int_0^y t^{-\frac{5}{2}}e^t\,dt\right)$. Eventually it would be necessary to include terms

$$e^{-\beta}(M_1(y)+M_2(y)/\beta+\ldots)\log\beta/\beta^2.$$

The initial values of the complementary functions T_1, Q_1 may be found approximately from the following relations which obtain for $y \ll 1$:

$$T_1 = 4(1-y+\tfrac{3}{8}y^2+\ldots),$$
$$Q_1 = \tfrac{2}{3}\log y+\tfrac{3}{16}y-\tfrac{205}{96}+\ldots$$

or $\sigma = \frac{1}{2}$ and

$$T_1 = -4(1-2y)(\log 2y + 1 + 2\gamma + \ldots),$$
$$Q_1 = -\tfrac{1}{3}\log^2 y + D\log y + E + Fy\log^2 y + \ldots,$$
$$B = -4(\log 2 + 1 + 2\gamma), \quad D = (3B+8)/18, \quad F = (8-3D)/21,$$
$$E = -\tfrac{1}{2}(B+4F+2D) \quad \text{for} \quad \sigma = 2.$$

Approximate temperature and drift-velocity profiles, obtained by keeping terms to $O(1/\beta)$ in the series solutions, are shown in figures 2, 3 for $\beta = 6$ ($x = 40{\cdot}17$); the profiles are expressed in terms of the variables x, y, t, and w of §4.

6. Equations of motion when buoyancy dominant

Because we are considering the approach to a steady-state régime independent of displacement from the leading edge and because our assumptions guarantee a thin boundary layer, (1)–(4) reduce to

$$\left.\begin{aligned} W_s + V_n &= 0, \\ VW_s + WW_n &= \nu W_{nn} + \beta g(T_w - T_\infty)\,\vartheta \sin(s/a), \\ W\vartheta_s + V\vartheta_n &= \kappa\vartheta_{nn}, \\ WU_s + VU_n &= \nu U_{nn} \end{aligned}\right\} \quad (12)$$

in physical variables. Here n is the normal measured outwards from the surface of the cylinder; s is arc length measured clockwise from the lowest generator (looking downstream); $\vartheta = (T - T_\infty)/(T_w - T_\infty)$. U is the velocity component corresponding to z, the axial displacement. V and W are the components of velocity associated with n and s, respectively. The boundary conditions are

$$U = V = W = 0, \quad \vartheta = 1 \quad \text{on} \quad n = 0,$$

and $$U_\infty = U, \quad V = V(s), \quad W = \vartheta = 0 \quad \text{as} \quad n \to \infty.$$

We can satisfy the first of (12) by introducing a stream function S and taking $W = S_n$, $V = -S_s$. Let $x = s/a$, $y = \mathscr{G}^{\frac{1}{4}}n/a$, $f = S/\nu\mathscr{G}^{\frac{1}{4}}$ so that $W = \nu\mathscr{G}^{\frac{1}{2}}f'a$, $V = -\nu\mathscr{G}^{\frac{1}{4}}f_x/a$. The rest of (12) become

$$\left.\begin{aligned} f'f'_x - f_x f'' &= f''' + \vartheta \sin(x), \\ f'\vartheta_x - f_x\vartheta' &= \vartheta''/\sigma, \\ f'U_x - f_x U' &= U'', \end{aligned}\right\} \quad (13)$$

with $U = f = f' = 0$, $\vartheta = 1$ on $y = 0$ and $U = 1, f = f(x)$, $\vartheta = 0$ as $y \to \infty$.

The first two of (13) have been obtained by Ostrach (1964).

7. The solution valid near the lowest generator

In order to get a solution which is a good approximation near the lowest generator of the cylinder (i.e. near $x = 0$) let

$$f(x,y) = f_1(y)\,x + f_3(y)\,x^3 + \ldots,$$
$$\vartheta(x,y) = t_0(y) + t_2(y)\,x^2 + \ldots,$$
$$U(x,y) = u_0(y) + u_2(y)\,x^2 + \ldots.$$

Inserting these into (13) and equating coefficients of like powers of x gives

$$\left.\begin{aligned} f_1''''+f_1f_1''-(f_1')^2+t_0 &= 0,\\ t_0''+f_1t_0'\sigma = 0, \quad u_0''+u_0'f_1 &= 0,\\ f_3''''+3f_1''f_3+f_3''f_1-4f_1'f_3'+t_2-\tfrac{1}{6}t_0 &= 0,\\ t_2''+\sigma(t_2'f_1-2t_2f_1'+3t_0'f_3) &= 0,\\ u_2''+u_2'f_1-2u_2f_1'+3u_0'f_3 &= 0,\end{aligned}\right\} \qquad (14)$$

using the series expansion of $\sin x$. The boundary conditions are $f_n(0) = f_n'(0) = 0$, $n = 1, 3$; $f_1(\infty) = \text{const.}$, $f_3(\infty) = 0$; $t_0(0) = 1$, $t_0(\infty) = 0$; $t_2(0) = t_2(\infty) = 0$; $u_n(0) = 0$, $n = 0, 2$; $u_0(\infty) = 1$, $u_2(\infty) = 0$.

Equations (14) for the f's and t's have been obtained by Poots (1964).

In this case they were solved numerically and used to provide initial profiles for the step-by-step technique discussed below in §8. The initial values for the integration of (14) are presented in table 3 for the cases $\sigma = \frac{1}{2}, 2$.

	$\sigma = \frac{1}{2}$	$\sigma = 2$
$f_1(0)$	0	0
$f_1'(0)$	0	0
$f_1''(0)$	0·89780	0·73290
$t_0(0)$	1	1
$t_0'(0)$	−0·32621	−0·53491
$u_0(0)$	0	0
$u_0'(0)$	0·45772	0·37999
$f_3(0)$	0	0
$f_3'(0)$	0	0
$f_3''(0)$	−0·09677	−0·08154
$t_2(0)$	0	0
$t_2'(0)$	0·01396	0·02432
$u_2(0)$	0	0
$u_2'(0)$	−0·00091	−0·00089

TABLE 3. Initial conditions for equations (14)

8. Stepwise continuation of the primarily convective solution

Once again the starting point is equations (13). Instead of forming the finite-difference counterparts of these we begin by making a Görtler-inspired transformation due to Saville & Churchill (1967) which has the advantage that the temperature and velocity profiles expressed in terms of the new variables change relatively little as x proceeds from zero to π. With

$$\xi = \int_0^x \sin^{\frac{1}{3}}(z)\,dz, \quad \eta = (\tfrac{3}{4}\xi)^{\frac{1}{4}} y \sin^{\frac{1}{3}}(x), \quad f(x,y) = (\tfrac{4}{3}\xi)^{\frac{3}{4}} F(\xi,\eta),$$

equations (13) become

$$\left.\begin{aligned} (\tfrac{4}{3}\xi)(F_\eta F_{\eta\xi}-F_\xi F_{\eta\eta})-F_{\eta\eta}F+\tfrac{4}{3}K(\xi)F_\eta^2 &= \vartheta+F_{\eta\eta\eta},\\ (\tfrac{4}{3}\xi)(F_\eta\vartheta_\xi-F_\xi\vartheta_\eta)-\vartheta_\eta F &= (1/\sigma)\vartheta_{\eta\eta},\\ (\tfrac{4}{3}\xi)(F_\eta U_\xi-F_\xi U_\eta)-U_\eta F &= U_{\eta\eta}.\end{aligned}\right\} \qquad (15)$$

$$K(\xi) = \tfrac{1}{2} + \frac{\xi}{3\sin x}\frac{d\sin x}{d\xi}; \quad K(\xi) = \tfrac{3}{4} - \tfrac{3}{40}(\tfrac{64}{27})^{\frac{1}{3}}\xi^{\frac{2}{3}} + \dots$$

when $\xi \ll 1$.

Equations (15) were transformed into finite-difference equations in the way described in Terrill's (1960) article and sketched briefly in §§ 2–5.

By analogy to Mitchell's (1961) use of a variable step-length to overcome the singularity in the von Mises equation for the flat plate, an exponentially increasing step was used for $d\eta$ to circumvent difficulties arising from the functions'

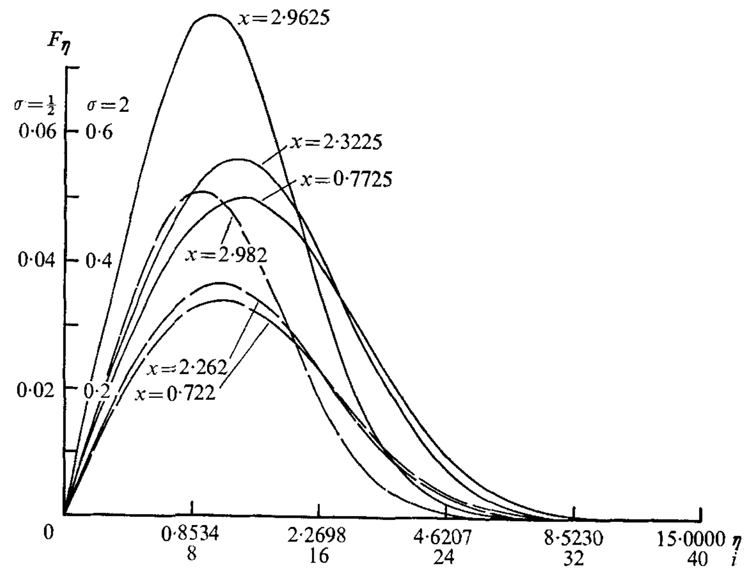

FIGURE 4. Drift velocity (stream function) profiles. ——, $\sigma = \frac{1}{2}$; — —, $\sigma = 2$.

slow $(e^{-\eta\sigma})$ approach to their final values as $\eta \to \infty$. (In the first half of the paper the corresponding variation is $e^{-y^2\sigma}$.) 'Infinity' conditions were imposed at $\eta = 15$; this value was reached after forty steps of length given by $d\eta_i = 1{\cdot}065388^{i-40}$.

Except that an exponentially growing mesh-length was used for η, the computational procedure was identical with that described earlier. The profiles for $x = 0$ were used as the initial profiles at $x = 0{\cdot}01$; the initial increment was $dx = 0{\cdot}0005$. The results were judged acceptable when two successive iterations produced agreement to within 5×10^{-5} everywhere. The latest iteration for F_η was used to obtain an improved result for ϑ and vice versa until the results for both functions were satisfactory.

Figures 4–6 show velocity and temperature profiles at a few values of x ranging from about $x = 0{\cdot}011$ to about $x = 3$. Table 4, which is available from the editor, contains sample values of the flow parameters $F_{\eta\eta}(\xi, 0)$, $\vartheta_\eta(\xi, 0)$, $U_n(\xi, 0)$, $F(\xi, \infty)$, $F_\xi(\xi, \infty)$, and dw, $d\vartheta$, and dU, where

$$dW = \int_0^\infty F_\eta^2\, d\eta, \quad d\vartheta = \int_0^\infty F_\eta \vartheta\, d\eta, \quad dU = \int_0^\infty F_\eta U\, d\eta.$$

The flow parameters were obtained by the methods described before; they are quoted for $\sigma = \frac{1}{2}, 2$.

As a check on accuracy, the profiles at $x = 0{\cdot}1355$ ($\xi = 0{\cdot}05225$) obtained by the step-by-step computations were compared with profiles obtained from the series method for the same value of x. The agreement was poorest in the last few values near $\eta = 15$; the largest discrepancies amounted to about $\pm 0{\cdot}001$.

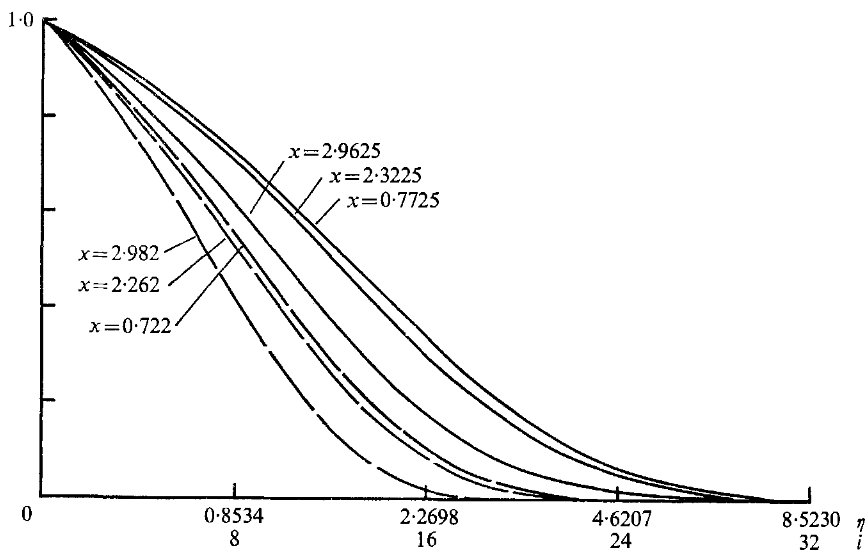

FIGURE 5. Temperature profiles. ——, $\sigma = \frac{1}{2}$, — —, $\sigma = 2$.

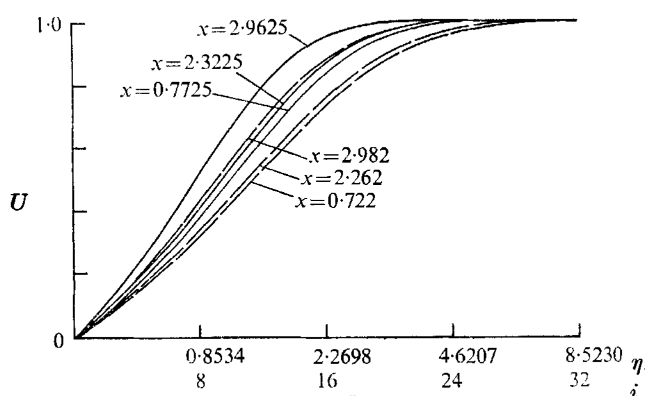

FIGURE 6. Axial velocity profiles. ——, $\sigma = \frac{1}{2}$; — —, $\sigma = 2$.

Of course this places a limit on the accuracy of the flow parameters. With a different adjustment of the variable mesh-length, i.e. with a shorter final step and probably a slightly larger number of steps to cover the interval, it should not be difficult to obtain results reliable to four decimal places. Another possible improvement would be the use of a slightly longer interval than $0 \leqslant \eta \leqslant 15$.

This article is an adaptation of my Ph.D. thesis (University of Manchester), which contains more complete numerical results for the three cases $\sigma = \frac{1}{2}, 1, 2$.

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